8 Heat Equation on the Real Line

8.1 General Solution to the 1D heat equation on the real line

From the discussion of conservation principles in Section 3, the 1D heat equation has the form

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \quad \text{on domain } |x| < \infty, t > 0. \tag{1}$$

The goal of this section is to construct a general solution to (1) for $x \in \mathbb{R}$, then consider solutions to initial value problems (Cauchy problems) involving the heat equation.

Exercises

1. By taking the appropriate derivatives, show that

$$S(x,t) = \frac{1}{2\sqrt{\pi D t}} e^{-x^2/4Dt} \tag{2}$$

   is a solution to (1). S is sometimes referred to as the *source function*, or *Green's function*, or *fundamental solution* to the heat equation.

2. Show that $S(x,t)$ in (2) also satisfies, for any fixed $t > 0$,

$$\int_{-\infty}^{\infty} S(x,t)\ dx = 1.$$

   (Hint: Make the change of variables $r = \frac{x}{2\sqrt{Dt}}$ and remember that $\int_0^\infty e^{-r^2}\ dr = \sqrt{\pi}/2$.) This result implies, for each $x \in \mathbb{R}$, for any $t > 0$, $\int_{-\infty}^{\infty} S(x-y,t)dy = 1$.

3. These exercises show that without further conditions, solutions to (1) can take many forms. Show that the function $u(x,t) = x^2 + 2t$ is a solution to (1) with $D = 1$. (This is one member of a whole family of multinomials (polynomials in x and t) that are solutions to the heat equation; they are called *heat polynomials*. However, they are not bounded at infinity, so they have limited use for us (or in typical applications).

4. Show that $u(x,t) = e^{-Dt}\sin(x)$ is a solution to (1).

5. What relationship exists between constants a and b such that $u(x,t) = e^{at}\cos(bx)$ is a solution to (1)?

Now we want to obtain a general solution to (1). As a strategy, the question is, can we reduce (1) to solving an ODE?[1] The answer is yes, through the

Boltzmann (similarity) transformation: Let

$$u(x,t) = f(\eta), \text{ where } \eta = x/\sqrt{Dt}. \tag{3}$$

Now

$$\frac{\partial u}{\partial t} = \frac{df}{d\eta}\frac{\partial \eta}{\partial t} = -\frac{\eta}{2t}\frac{df}{d\eta},$$

and

$$\tfrac{\partial u}{\partial x} = \tfrac{df}{d\eta}\tfrac{\partial \eta}{\partial x} = \tfrac{1}{\sqrt{Dt}}\tfrac{df}{d\eta}, \text{ hence } \tfrac{\partial^2 u}{\partial x^2} = \tfrac{1}{Dt}\tfrac{d^2 f}{d\eta^2}$$

so substituting these into (1) gives

$$-\frac{\eta}{2}\frac{df}{d\eta} = \frac{d^2 f}{d\eta^2}, \tag{4}$$

an ODE for f. Not only that, it is linear, first-order equation in $df/d\eta$,

$$\frac{d}{d\eta}(\frac{df}{d\eta}) + \frac{\eta}{2}(\frac{df}{d\eta}) = 0,$$

which has $e^{\eta^2/4}$ as an integrating factor. Multiplying (4) by this exponential means we can write (4) as

$$\tfrac{d}{d\eta}(e^{\eta^2/4}\tfrac{df}{d\eta}) = 0, \text{ or } e^{\eta^2/4}\tfrac{df}{d\eta}(\eta) = constant = C_0$$

so

$$\begin{aligned} f(\eta) &= C_0\int_0^{\eta} e^{-\bar{\eta}^2/4}\, d\bar{\eta} + C_2 \\ &= 2C_0\int_0^{\eta/2} e^{-s^2}\, ds + C_2. \quad (s = \bar{\eta}/2) \end{aligned}$$

[1] By following our Principle 1 in section 00

(Remember, with a variable limit **always** use a 'dummy' variable of integration.) Therefore, with $C_1 = 2C_0$,

$$u(x,t) = C_1 \int_0^{x/2\sqrt{Dt}} e^{-s^2} ds + C_2. \tag{5}$$

This is the **general solution to (1)** we were looking for.

Remark: The physical diffusivity parameter D has dimensional units of $length^2/time$, so $\sqrt{Dt}$ has dimensional units of length; hence η, as defined by (3), is *dimensionless*, being a ratio of length/length = (dimension of x)/(dimension of $\sqrt{Dt}$). Note also that for the heat equation the change of variables $u(\tilde{x},\tilde{t}) = u(\alpha x, \alpha^2 t)$ leaves the equation invariant, suggesting to us to consider the *similarity* variable $\alpha x/\sqrt{\alpha^2 t} = x/\sqrt{t}$. For the wave equation $w_{tt} = w_{xx}$, the change in variables $w(\tilde{x},\tilde{t}) = w(\alpha x, \alpha t)$ leaves the equation invariant, suggesting we try the same reduction as the heat equation, but with $\eta = x/t$.

Exercises

1. Show that the only substitution $u = f(\eta)$, with η of the form $\eta = xt^\beta$, into (1), and reducing the problem to an ordinary differential equation, is when $\beta = -1/2$.
   (Remember, we must end up with coefficients in the f equation only in terms of η, not with x or t.)

2. Consider the Boltzmann transformation (3) for the vibrating string equation $\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2$; that is, assume $u(x,t)$ has the form $f(\eta)$, with $\eta = xt^{-1/2}$. Show that we can **not** reduce the string equation to an ODE with this transformation. However, show we can with the substitution $\eta = xt^{-1}$.

Remark: In the review section on ODEs in the appendix the *error function*

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-r^2} dr$$

is introduced. If we let $C_3 := C_1\sqrt{\pi}/2$, we can write (5) in terms of the error function; that is,

$$u(x,t) = C_3 erf(\frac{x}{2\sqrt{Dt}}) + C_2. \tag{6}$$

Example: Consider the problem

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \text{ on } |x| < \infty,\ t > 0$$

$$u(x,0) = H(x)$$

Here $H(\cdot)$ is the Heaviside function introduced in Section 5.

For $x < 0$

$$\begin{aligned} 0 = u(x,0+) &= \lim_{t\to 0+}\{C_1 \int_0^{x/2\sqrt{Dt}} e^{-s^2} ds + C_2\} \\ &= C_1 \int_0^{-\infty} e^{-s^2} ds + C_2 \quad (r = -s) \\ &= -C_1 \int_0^{\infty} e^{-r^2} dr + C_2 = -C_1\frac{\sqrt{\pi}}{2} + C_2. \end{aligned}$$

Also, for $x > 0$,

$$\begin{aligned} 1 = u(x,0+) &= \lim_{t\to 0+}\{C_1 \int_0^{x/2\sqrt{Dt}} e^{-s^2} ds + C_2\} \\ &= C_1 \int_0^{\infty} e^{-s^2} ds + C_2 = C_1\frac{\sqrt{\pi}}{2} + C_2. \end{aligned}$$

Since $C_2 = C_1\sqrt{\pi}/2$, then $C_2 = 1/2$ and $C_1 = 1/\sqrt{\pi}$, so that

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_0^{x/2\sqrt{Dt}} e^{-s^2} ds + \frac{1}{2} = \frac{1}{2}\{1 + erf(\frac{x}{2\sqrt{Dt}})\} \tag{7}$$

Exercises

1. Show that $u_1(x,t) = e^{-t}\cos(x)\cos(\frac{1}{2}\tan(x))$ and $u_2(x,t) = e^{-t}\cos(x)\sin(\frac{1}{2}\tan(x))$ are solutions to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + q(x)u, \text{ where } q(x) = \frac{1}{4}\sec^4(x)$$

2. In the following problem find constants α, β such that the function substitution $u(x,t) = e^{\alpha x - \beta t} w(x,t)$ reduces the given diffusion-advection equation to the heat equation for w:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - 6\frac{\partial u}{\partial x} - 2u$$

3. For a small dendritic branch of a nerve cell the membrane potential satisfies an equation of the form

$$C_m \frac{\partial v}{\partial t} + g(v - E) = \frac{a}{2R_i}\frac{\partial^2 v}{\partial x^2}$$

where the constants C_m, g, E, a, R_i represent, respectively, membrane capacitance, conductance, reversal potential, fiber radius, and axoplasmic resistivity. Let $\tilde{v}(\tilde{x}, \tilde{t}) = v(x,t)/E - 1, \tilde{x} = x/\lambda, \tilde{t} = t/\tau$. What must λ and τ be in terms of the original constants such that $\tilde{v}$ satisfies

$$\frac{\partial \tilde{v}}{\partial \tilde{t}} + \tilde{v} = \frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2}?$$

8.2 The Cauchy problem for the 1D heat equation

The Cauchy problem (or initial-value problem, IVP) for the heat equation on the domain $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ constitutes the equation (1) along with a prescribed initial condition; that is, a specified initial heat distribution, $u(x,0) = f(x)$ at some point in time we call $t = 0$. The function just needs to be defined on $\mathbb{R}$; it does not need to be continuous, though it is usually considered integrable on $\mathbb{R}$. We will assume f to be at least piecewise continuous, that is, continuous except at a finite number of points.

We derive the general formula for the solution to the Cauchy problem in this section. Our aim here is to reason through what the solution form should be in an informal way. In a later section on transform methods, we'll employ the Fourier transform to obtain the solution formula as an application of the transform.

If $u(x,t)$ satisfies the heat equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}\,, \quad \text{on } |x| < \infty,\ t > 0, \tag{8}$$

then so does $w(x,t) = \frac{\partial u}{\partial x}(x,t)$ (just differentiate the equation). By direct differentiation of (7) (using the Leibniz rule),

$$w(x,t) = \frac{\partial u}{\partial x}(x,t) = \frac{e^{-x^2/4Dt}}{2\sqrt{\pi Dt}},$$

and this is just the fundamental solution $S(x,t)$ mentioned in the first exercises of section 8.1. Note that, for fixed $t > 0$, w graphs as a bell-shaped graph, symmetric about $x = 0$, and satisfies $\int_{-\infty}^{\infty} w(x,t)dx = 1$. As a solution to (8), think of $w(x,t)$ as the temperature that results at x at time t from an initial unit heat source at the origin at time $t = 0$. Because of the constant coefficient equation, the shifting of the temperature profile again leads to a solution of the heat equation; that is, $w(x-y,t)$ is the temperature at x at time t caused by a unit heat source at y at time $t = 0$. If $f(y)$ is a function that represents the magnitude of the heat source at location y (y is anywhere on the real line), then $f(y)w(x-y,t)$ gives the resulting temperature at x at time t due to a heat source of magnitude $f(y)$ at location y given at time $t = 0$. Since we have distributed heat sources over the whole real line, the temperature at x at time t should be the accumulated effects of all sources; that is,

$$u(x,t) = \int_{-\infty}^{\infty} f(y)w(x-y,t)dy = \frac{1}{2\sqrt{\pi Dt}} \int_{-\infty}^{\infty} f(y)e^{-(x-y)^2/4Dt}dy. \quad (9)$$

Exercise: Assuming you can interchange differentiation and integration, show that $u(x,t)$, defined by (9), actually solves the heat equation (8). (This is really a consequence that S, hence $w(x-y,t)$, solves the heat equation.)

Given that f is a continuous function, then for each fixed x, $f(x - 2r\sqrt{Dt}) \to f(x)$ as $t \to 0+$. Let $r := \frac{x-y}{2\sqrt{Dt}}$, then $2\sqrt{Dt}\,dr = -dy$, so then (9) can be written as

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x - 2r\sqrt{Dt})e^{-r^2}dr.$$

Thus, as $t \to 0$,

$$u \to \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x)e^{-r^2}dr = f(x)\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-r^2}dr = f(x)\frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-r^2}dr = f(x).$$

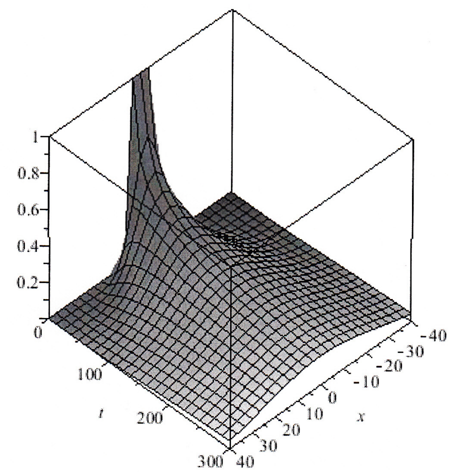

Figure 1: Solution to heat equation with $D = 1$, $u(x, 0) = H(5 - |x|)$.

(Recall that e^{-r^2} is an even function and that $\int_0^\infty e^{-r^2} dr = \sqrt{\pi}/2$.)
The only non-rigorous part of the argument is that we did not justify the need to interchange integration with the limit, which is doable under very mild conditions, but requires a bit more real analysis development than we will not pursue here. So, modulo a couple of technical details, we have that $u(x, t)$, as given by (9), is the solution to the problem

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$

$$u(x, 0) = f(x)$$

defined on the domain $|x| < \infty$, $t > 0$.

Example 1: Let $f(x) = H(a - |x|)$; that is, $f(x) = 1$ if, and only if $-a < x < a$ ($a > 0$), and zero otherwise.

Although the initial condition is discontinuous, the diffusion process will instantly smooth it out. (You can see this, e.g., in figure 1. Integration is a smoothing operation, and kernel S is integrable and infinitely differentiable.)

By (9), using the above change of variables,

$$u(x,t) = \frac{1}{2\sqrt{\pi Dt}} \int_{-a}^{a} e^{-(x-y)^2/4Dt} dy = \frac{1}{\sqrt{\pi}} \int_{\frac{x-a}{2\sqrt{Dt}}}^{\frac{x+a}{2\sqrt{Dt}}} e^{-r^2} dr$$

$$= \frac{1}{2}\{\frac{2}{\sqrt{\pi}} \int_{\frac{x-a}{2\sqrt{Dt}}}^{0} e^{-r^2} dr + \frac{2}{\sqrt{\pi}} \int_{0}^{\frac{x+a}{2\sqrt{Dt}}} e^{-r^2} dr\}$$

$$= \frac{1}{2}\{erf(\frac{x+a}{2\sqrt{Dt}}) - erf(\frac{x-a}{2\sqrt{Dt}})\}$$

See figure 1, $D = 1, a = 5$.

Example 2: Consider the IVP (motivated by neutron transport theory)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - ku$$

$$u(x,0) = f(x)$$

for $k > 0$ constant, and $-\infty < x < \infty, t > 0$. By letting $u(x,t) = e^{-kt}v(x,t)$, then substituting into this problem, $v(x,t)$ satisfies

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$$

$$v(x,0) = u(x,0) = f(x),$$

so, from (9),

$$v(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(y)e^{-(x-y)^2/4t} dy \Rightarrow u(x,t) = \frac{e^{-kt}}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(y)e^{-(x-y)^2/4t} dy.$$

Remark: This is an example of Principle 2 expressed at the end of the Introductory Section 00. The $-ku$ term acts like a sink (or a dissipative term, or death term), depending on the physical context.

Summary: From this section you need to remember the fundamental solution of the heat equation, the general solution of the heat equation, and the solution of the Cauchy problem for the 1D heat equation on the real line. It is worth keeping in mind the change of variables that simplifies the exponent in the heat kernel that was used in this subsection.

Exercises:

1. Solve

$$\frac{\partial v}{\partial t} = D\frac{\partial^2 v}{\partial x^2} \text{ on } |x| < \infty,\ t > 0$$

$$v(x,0) = 2x \text{ on } |x| < \infty$$

2. Consider the simple advection-diffusion equation

$$\frac{\partial u}{\partial t} + \mu\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + F$$

where the speed μ and source term F are constants. The constant term can be absorbed by a change of variable. Let $v = u - Ax$. What is A so that v satisfies $v_t + \mu v_x = v_{xx}$? If we instead let $v = u - Bt$, what would B be to achieve the same result?

3. Show that the advection-diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - a\frac{\partial u}{\partial x} - bu \text{ on } |x| < \infty,\ t > 0,$$

where a, b, D are positive constants, can be transformed to the heat equation for $v(x,t)$ if we let $u(x,t) = e^{\alpha x - \beta t}v(x,t)$. Determine α, β in terms of a, b, D to do this.
(ans: $\alpha = a/2D, \beta = b + a^2/4D$.)

4. (a) Show that the nonlinear equation $u_t = u_{xx} + (u_x)^2$ can be reduced to the heat equation by a change in dependent variable $w = e^u$. (It is a Big Deal when a transformation can be found that changes a nonlinear equation to a linear one!)

   (b) With the nonlinear diffusion equation for u in part (a) being defined for $|x| < \infty$, $t > 0$, and with $u(x,0) = f(x), |x| < \infty$, using part (a), write out the formula of the solution $u(x,t)$.
(Ans: $u(x,t) = \ln(\int_{-\infty}^{\infty} S(x-y,t)e^{f(y)}dy)$, where S is the fundamental solution to the heat equation.)

5. Here is a neat observation. Consider the heat equation in 3-space: $u_t = D\nabla^2 u = D\{u_{xx} + u_{yy} + u_{zz}\}$. (Appendix C has a derivation of the heat equation in higher dimensions that just extends the argument given for the derivation in one space dimension as presented in section 3.) If we considered the Laplace operator ∇^2 in terms of spherical coordinates, then

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin\theta}\frac{\partial^2}{\partial \phi^2}.$$

(This is an important operator in the study of geophysical flow, e.g., but is not particularly nice to work with at this stage.) However, if we impose the condition of **spherical symmetry**, u would be independent of the two coordinate angles, so we only have to consider the *radial part of the operator*, that is $\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})$. Thus, we consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{D}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}) \quad \text{for } r > 0, t > 0. \tag{10}$$

If $v(r,t) = ru(r,t)$, where u satisfies (10), show that v solves the (ordinary) heat equation

$$\frac{\partial v}{\partial t} = D\frac{\partial^2 v}{\partial r^2} \quad .$$

(Unfortunately there is no corresponding transformation in the 2D radially symmetric case.)

6. Burger's equation,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

comes up in the study of gas dynamic situations. Show that the transformation $u(x,t) = -2\frac{\partial}{\partial x}\ln(v(x,t))$ give the solution to Burger's equation if $v(x,t)$ satisfies the heat equation $v_t = v_{xx}$.
(This is the Cole-Hopf transformation, and it is one of the first nontrivial cases where an important nonlinear PDE was able to be transformed to a linear PDE; so it was a Big Deal.)